1.6. The Natural Logarithm

Motivation: Population in Nevada, t years since 2000 is given by

$$P = f(t) = 2.020(1.036)^t \text{ millions}$$

Ques. How do we find when the population will reach 4 million?

We want t such that

$$f(t) = 4$$

$$2.020(1.036)^t = 4$$

Not obvious how to solve for t.

We will take log. on both sides:

We will comb back to this.

Def. The **natural logarithm of x**, written $\ln(x)$, is the power/exponent of e needed to get x.

e = Euler's number = $2.71828\ldots$

$$\ln(x) = c \quad \text{means}$$

$$e^c = x$$

Also written as $\log_e(x)$. It is the inverse of e^x.

$$\begin{tikzcd} x \arrow[r, "\exp"] & e^x \arrow[l, bend left, "\ln"] \arrow[r, "\ln"] & \ln(e^x) = x \end{tikzcd}$$

$$\begin{tikzcd} 4 \arrow[r, "\exp"] & e^4 \arrow[l, bend left, "\ln"] \end{tikzcd} \qquad \ln(e^4) = 4$$

Ex. (i) $\ln(e) = ?$

$$e^{?} = e$$

$$? = 1$$

$$\therefore \ln(e) = 1$$

(ii) $\ln(e^5) = ?$

$$e^{?} = e^5$$

$$? = 5$$

$$\therefore \ln(e^5) = 5$$

(iii) $\ln(1) = ?$

$$e^{?} = 1$$

We know $e^0 = 1$, So, $? = 0$

$$\therefore \ln(1) = 0.$$

$$\ln(x)$$

$$e^{?} = x$$

(iv) $\ln(0)$

$$e^{?} = 0$$

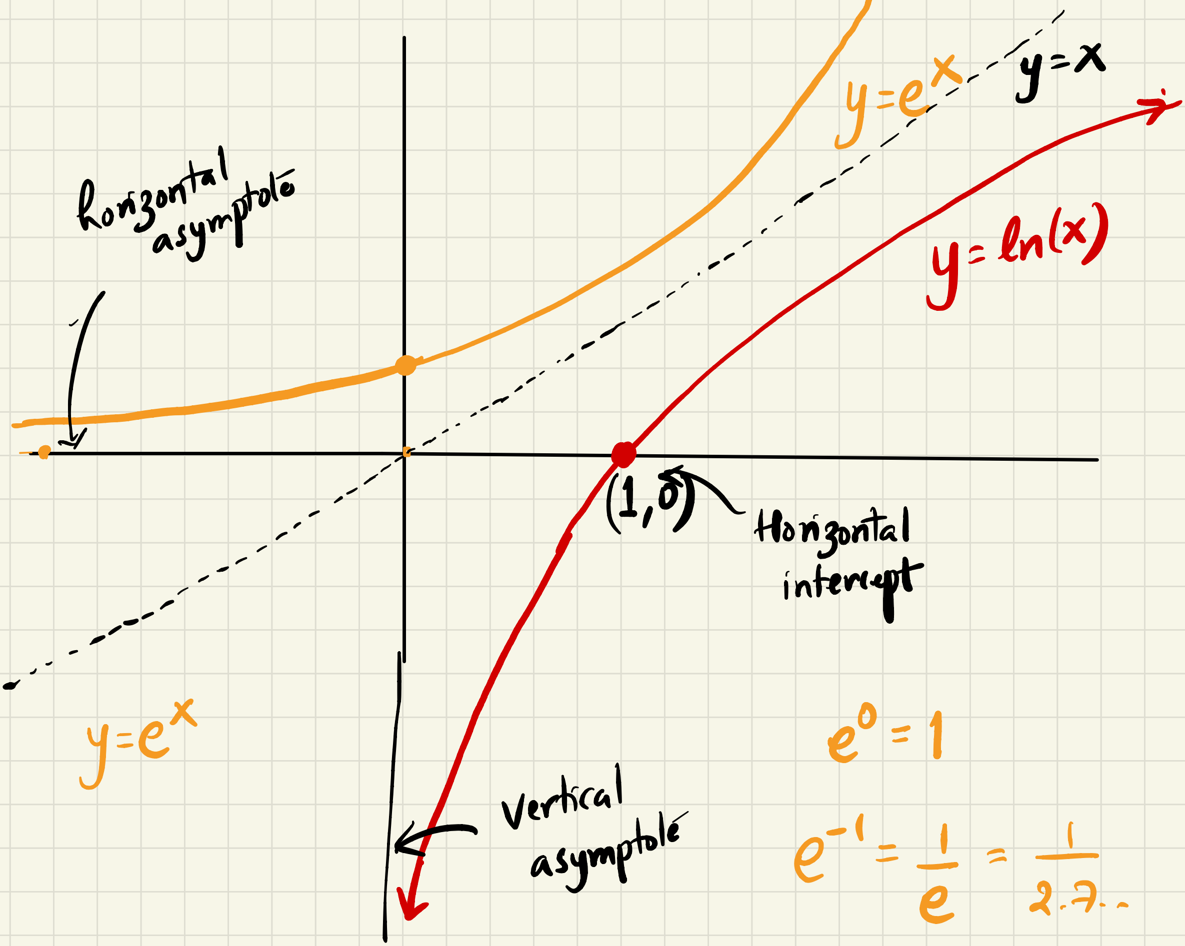
Cannot be 1

because $e^1 = e$

Cannot be 0

because $e^0 = 1$

<u>undefined</u>



$e^0 = 1$

$e^{-1} = \frac{1}{e} = \frac{1}{2.7..}$

< 1

$e^{-10} = \frac{1}{e^{10}} =$ small no.

For What value of x is

$\ln(x) = 0$

$\ln(1) = 0$

Properties of Natural Logarithms

1) $\ln(AB) = \ln A + \ln B$ (Products into sums)

Warning: $\ln(A+B) \neq \ln A + \ln B$

2) $\ln\left(\frac{A}{B}\right) = \ln(A) - \ln(B)$ (Fractions / Quotient into difference)

Warning: $\ln(A-B) \neq \ln(A) - \ln B$

3) $\ln(A^p) = p\ln(A)$ (bring down exponents / powers)

4) $\ln(e^x) = x$

5) $e^{\ln x} = x$

Problem 1

Find t such that $3^t = 10$

Soln.

$$3^t = 10$$

$$\ln(3^t) = \ln(10)$$

$$t \ln(3) = \ln(10) \qquad [\text{ Property } 3]$$

$$\frac{t \cancel{\ln(3)}}{\cancel{\ln(3)}} = \frac{\ln(10)}{\ln(3)}$$

$$t = \frac{\ln(10)}{\ln(3)}$$

$$\boxed{t = 2.096}$$

$$3^t = 10$$

Take $\log_{10}$ on both sides

$$\log_{10}(3^t) = \log_{10}(10)$$

$$t \log_{10}(3) = \log_{10}(10)$$

$$t \log_{10}(3) = 1$$

$$t = \frac{1}{\log_{10}(3)}$$

$$\approx 2.096$$

Exponential Functions with Base e

$$P = P_0 a^t$$

Goal: Rewrite in terms of base e

We want to do that, taking ln is computationally easier.

$$P = P_0 a^t$$

Let $a = e^k$ for some k.

$$\Rightarrow P = P_0 (e^k)^t$$

$$\therefore P = P_0 e^{kt}$$

We succeeded in our goal

What is this k?

$$a = e^k$$

So, $\boxed{k = \ln(a)}$

$P = P_0 a^t$ exponential function

Rewrite as $P = P_0 e^{kt}$

where $k = \ln a$

is called continuous growth or decay rate.

- If $a > 1$, we have exponential growth.
  If $0 < a < 1$ we have exponential decay.

- If $k > 0$, we have exponential growth.
  If $k < 0$, we have exponential decay.

$$k = \ln a$$

$$a > 1$$

equivalent

$$k = \ln(a) > 0$$

$$0 < a < 1$$

equivalent

$$\Leftrightarrow \quad k = \ln(a) < 0$$

Problem 4

a) Convert $P = 1000e^{0.05t}$ to the form $P = P_0 a^t$

Soln. $P = 1000e^{0.05t}$

$K = 0.05$

We know,

$K = \ln a$

$0.05 = \ln(a)$

$\ln(a) = 0.05$ By definition

$e^{0.05} = a$

$\therefore a = 1.0513$

$\therefore P = 1000(1.0513)^t$

So a continuous growth rate of 5% is equal to growth rate of 5.13%

$a = 1 + r$
$r = a - 1$

b) Convert $P = 500(1.06)^t$ to the form

$P = P_0 e^{kt}$

<u>Soln.</u>

$P_0 = 500$

$K = \ln a$

$K = \ln(1.06)$

$\underline{K = 0.0583}$

$\therefore \quad P = 500\, e^{0.0583t}$

So growth rate of 6% per unit time is equivalent to continuous growth of 5.83%.

1. The problems you need to solve is on Moodle

2. HW 2 Due Feb 2 Tuesday

3. Next Class in person

4). Sample Test Due Feb 9 midnight.